Incoherent self-similarities of the coupled amplified nonlinear Schrödinger equations

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Self-similar propagation in a system of coupled amplified nonlinear Schrödinger equations is studied. We find that each individual amplified nonlinear Schrödinger equation can sustain a component similariton with a quadratic phase, which is the asymptotic self-similar solution of the corresponding equation. Under a width-matching condition, the incoherent summation of all the component similaritons leads to another similariton with parabolic profile. Numerical simulations show that this incoherent parabolic similariton maintains all the characteristics of its coherent counterpart.

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Recently the discovery of incoherent solitons [1,2] has revived the investigation of coupled nonlinear Schrödinger-like equations [3–12]. Among them, coupled nonlinear Schrödinger equations (CNLSEs) have been extensively studied since the self-trapping of light beams in a slow Kerr-type nonlinear medium is well characterized by those equations [6–12]. As a relatively established subject, CNLSEs describe many interesting and important physical phenomena, which include multicomponent Bose-Einstein condensates (BECs) [13], temporal incoherent solitons [14,15], and nonlinear interactions of optical waves with different polarizations or with different wavelength [16].

The incoherent soliton solution of CNLSEs can be regarded as the extension of the coherent soliton governed by a single nonlinear Schrödinger equation (NLSE). It has been well known for decades that the anomalous dispersion and Kerr nonlinearity of an optical fiber accommodate temporal solitons. Recently, another temporal self-similar propagation solution, the parabolic similariton, has been observed in a fiber amplifier with normal dispersion and has been intensively studied both theoretically and experimentally [17–23]. As the asymptotic self-similar solution of the amplified NLSE, the similariton is formed and maintained due to the interplay of normal dispersion, Kerr nonlinearity, and gain, which transforms an arbitrary input pulse into an amplified, linearly chirped pulse with a parabolic temporal profile. The temporal parabolic similariton has found important applications in high-power fiber amplifiers and lasers [21–23].

Based on the strong analogies between the diffraction of paraxial optical beams and the dispersive propagation of quasi-monochromatic pulses in dielectric media, the spatial parabolic similariton, i.e., parabolic beam, has been proposed and demonstrated theoretically for the amplified NLSE [24]; interestingly (and in contrast to solitons), self-similar propagation is possible in $(2+1)$ dimensions. Besides nonlinear optics, such a parabolic similariton has also been predicted in the growth of BECs [25]. Since the concept of solitons has now been extended to include incoherent solitons, it is of interest to ask whether an incoherent parabolic similariton (IPS) can exist in amplified CNLSEs. In this paper, we show theoretically that such an excitation indeed exists.

The N-component amplified CNLSEs incorporating a linear gain term g have the form

$$\frac{\partial \varepsilon_j}{\partial \varsigma} + ia\frac{\partial^2 \varepsilon_j}{\partial \xi^2} = i\gamma\left(\sum_{m=1}^{N} |\varepsilon_m|^2\right)\varepsilon_j + \frac{g}{2}\varepsilon_j. \quad (1)$$

In nonlinear optics, Eq. (1) describes the propagation of pulses through a fiber amplifier or of a cw incoherent paraxial beam in a dielectric planar waveguide amplifier with a Kerr nonlinearity, and ε_j is the electric field envelope. The first term appearing on the right side of the Eq. (1) denotes the incoherent coupling among N components due to the intrinsic nonlinearity. The second-order derivative term accounts for group velocity dispersion (GVD) for the temporal pulse or geometrical diffraction for the spatial beam. In the context of the growth of multicomponent BECs, ε_j represents the wave-function description of the condensate so that $|\varepsilon_j|^2$ is the particle number density. The second term on the left side of Eq. (1) corresponds to the kinetic energy contribution. We start by considering a more general two-component temporal amplified CNLSE of the form

$$\frac{\partial \Psi_s}{\partial z} + i\frac{\beta_{2s}}{2}\frac{\partial^2 \Psi_s}{\partial T^2} = i\gamma_s[|\Psi_s|^2 + |\Psi_p|^2]\Psi_s + \frac{g_s}{2}\Psi_s, \quad (2)$$

$$\frac{\partial \Psi_p}{\partial z} + i\frac{\beta_{2p}}{2}\frac{\partial^2 \Psi_p}{\partial T^2} = i\gamma_p[|\Psi_p|^2 + |\Psi_s|^2]\Psi_p + \frac{g_p}{2}\Psi_p, \quad (3)$$

where Ψ_s and Ψ_p are the slowly varying envelopes associated with two mutually incoherent pulses (in the following, we call them s-pulse and p-pulse to make a distinction). Here incoherent means that there is no interference between such two pulses, and therefore, the total power of the two pulses is a direct sum of the powers of both pulses, i.e., $P_{\text{tot}}=|\Psi_s|^2+|\Psi_p|^2$. In contrast, this number would be $P_{\text{tot}}=|\Psi_s+\Psi_p|^2$ for two coherent pulses. β_{2j}, γ_j, and g_j denote group velocity dispersion coefficient, the nonlinear parameter, and the constant gain, respectively, for the s-pulse $(j=s)$ and p-pulse $(j=p)$. Multiplying Eq. (2) by Ψ_s^*, subtracting from the complex conjugate of the same equation, and integrating both sides of this expression yields

$$\int_{-\infty}^{\infty} |\Psi_s(z,T)|^2 dT = \exp(g_s z)\int_{-\infty}^{\infty} |\Psi_s(0,T)|^2 dT.$$

We can conclude that the total power in each pulse increases exponentially with propagation distance. There is no power exchange between the two pulses since the coupling terms only affect the phase of the envelopes. First, we would like to discuss the case with $g_p \neq g_s$. Without losing generality, we assume $g_s > g_p$. Because of the exponential growth of the pulse power, the s-pulse will overtake the p-pulse, given enough propagation distance. The total power P_{tot} of the incoherent pulses will be dominated by s-pulse, and the contribution from p-pulse will be negligible, which means the coupling term in Eq. (1) eventually disappears. Then Eq. (1) will become a single amplified NLSE, and the s-pulse will evolve into a parabolic similariton. Evidently, the power summation of the s- and p-pulses approaches the parabolic similariton asymptotically. It is obvious that the IPS exists for the case that two incoherent pulses experience different amplification. In the rest of this paper, we assume $g_p = g_s = g$.

Ψ_s and Ψ_p can be separated into their real amplitudes and phases to be $\Psi_s(z,T) = A_s(z,T)\exp[i\phi_s(z,T)]$ and $\Psi_p(z,T) = A_p(z,T)\exp[i\phi_p(z,T)]$, which transforms (2) and (3) into the following coupled equations in A_s, A_p, ϕ_s, and ϕ_p:

$$\frac{\partial\phi_s}{\partial z} = -\frac{\beta_{2s}}{2A_s}\frac{\partial^2 A_s}{\partial T^2} + \gamma_s P_{\text{tot}} + \frac{\beta_{2s}}{2}\left(\frac{\partial\phi_s}{\partial T}\right)^2 \tag{4a}$$

$$\frac{\partial A_s^2}{\partial z} = \beta_{2s}\frac{\partial}{\partial T}\left(A_s^2\frac{\partial\phi_s}{\partial T}\right) + gA_s^2 \tag{4b}$$

$$\frac{\partial\phi_p}{\partial z} = -\frac{\beta_{2p}}{2A_p}\frac{\partial^2 A_p}{\partial T^2} + \gamma_p P_{\text{tot}} + \frac{\beta_{2p}}{2}\left(\frac{\partial\phi_p}{\partial T}\right)^2 \tag{5a}$$

$$\frac{\partial A_p^2}{\partial z} = \beta_{2p}\frac{\partial}{\partial T}\left(A_p^2\frac{\partial\phi_p}{\partial T}\right) + gA_p^2. \tag{5b}$$

On the right side of Eqs. (4a) and (5a), there are three terms that contribute to the phase change of two pulses. Following the similar procedure of Ref. [24], a parameter $N_j(z,T)$ is introduced to compare the strength of the first two terms for each equation:

$$N_j^2(z,T) = \left|\gamma_j P_{\text{tot}} \middle/ \left(\frac{\beta_{2j}}{2A_j}\frac{\partial^2 A_j}{\partial T^2}\right)\right| = \frac{2\gamma_j}{\beta_{2j}}\left|A_j P_{\text{tot}} \middle/ \frac{\partial^2 A_j}{\partial T^2}\right|, \quad j = s,p.$$

Under the assumption $N_j^2 \gg 1$, the first term on the right-hand side of Eqs. (4a) and (5a) can be neglected to give

$$\frac{\partial\omega_s}{\partial z} = -\frac{\partial}{\partial T}\left[\gamma_s P_{\text{tot}}(z,T) + \frac{\beta_{2s}}{2}\omega_s^2\right] \tag{6a}$$

$$\frac{\partial A_s^2}{\partial z} = -\beta_{2s}\frac{\partial}{\partial T}(A_s^2\omega_s) + gA_s^2 \tag{6b}$$

$$\frac{\partial\omega_p}{\partial z} = -\frac{\partial}{\partial T}\left[\gamma_p P_{\text{tot}}(z,T) + \frac{\beta_{2p}}{2}\omega_p^2\right] \tag{7a}$$

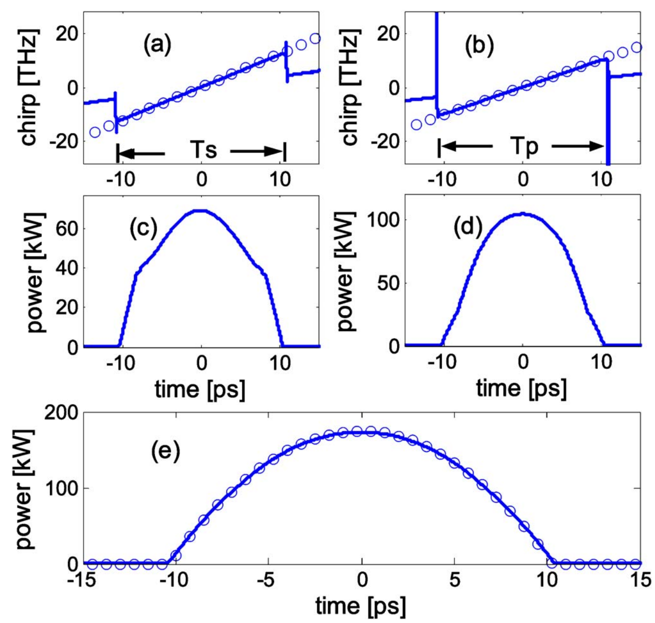

FIG. 1. (Color online) Incoherent parabolic similariton confirmed by numerical results after a propagation of 20 m. (a, b) chirps for s-pulse and p-pulse, respectively, (c, d) power profiles $|\Psi_s|^2$ and $|\Psi_p|^2$, and (e) parabolic profile of the total power defined by $P_{\text{tot}} = |\Psi_s|^2 + |\Psi_p|^2$. Solid curves represent the numerical results. Theoretical predictions are shown as circle curves in (a), (b), and (e). Although the results in the figures are for time-domain pulse propagation, they also could correspond to 1D beam profiles of diffraction in a nonlinear waveguide.

$$\frac{\partial A_p^2}{\partial z} = -\beta_{2p}\frac{\partial}{\partial T}(A_p^2\omega_p) + gA_p^2 \tag{7b}$$

where the chirp functions $\omega_s = -\partial\phi_s/\partial T$ and $\omega_p = -\partial\phi_p/\partial T$ have been introduced. As is well known, the asymptotic parabolic similariton obtained from a single amplified NLSE has a linear chirp $\omega(T) = (g/3\beta_2)T$, which is independent of nonlinear parameter γ. On the other hand, the coupling terms only modify the nonlinear parts of the coupled equations. It is a reasonable expectation that each pulse would still have a linear chirp $\omega_j(T) = (g/3\beta_{2j})T$, $(j = s,p)$ during the propagation. Substituting $\omega_j(T)$, $(j = s,p)$ into Eqs. (6b) and (7b) and adding yields

$$\frac{\partial P_{\text{tot}}(z,T)}{\partial z} = -\frac{g}{3}\frac{\partial}{\partial T}[P_{\text{tot}}(z,T)T] + gP_{\text{tot}}(z,T). \tag{8}$$

$P_{\text{tot}}(z,T)$ can be solved from Eqs. (6a), (7a), and (8). However, these three equations are more than enough to determine the parabolic similariton solution since there are only two unknown parameters, i.e., peak power and effective width. As a matter of fact, a combination of Eqs. (6a) and (8) or (7a) and (8) will give two solutions with the same peak power but generally different effective width. Evidently, these two solutions will become one consistent solution if they share the same effective width, which leads to the following IPS given $\beta_{2s}\gamma_s > 0$ and $\beta_{2p}\gamma_p > 0$:

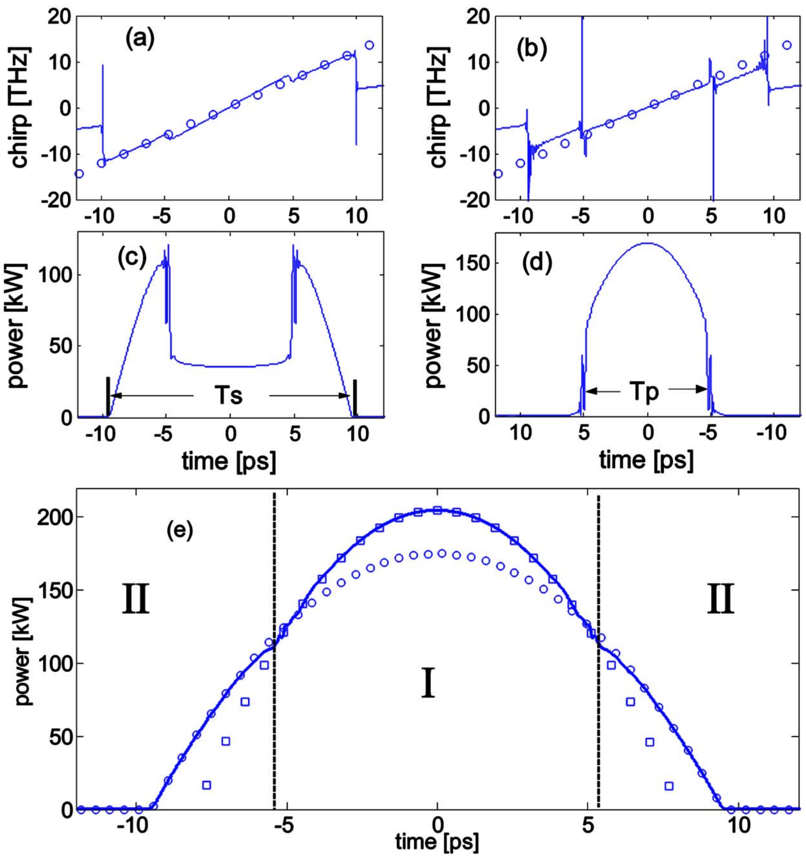

FIG. 2. (Color online) Numerical results after a propagation of 20 m with the violation of the width-matching condition. (a, b) chirps for s-pulse and p-pulse, respectively, and (c, d) power profiles $|\Psi_s|^2$ and $|\Psi_p|^2$. Solid curves represent the numerical results. Theoretical predictions are shown as circle curves in (a) and (b). (e) Piecewise parabolic fit of the total power profile $P_{\text{tot}}=|\Psi_s|^2+|\Psi_p|^2$. Regions I and II are well fit by two different parabolic function represented by square curve and circle curve, respectively.

$$P_{\text{tot}}(z,T) = P_0 \exp\left(\frac{2g}{3}z\right)\left[1 - \frac{T^2}{T_{pa}^2(z)}\right] \tag{9}$$

at $|T| \leqslant T_{\text{pa}}(z)$, where $P_{\text{tot}}(z,T)=0$ for $|T|>T_{\text{pa}}(z)$ with

$$T_{pa}(z) = \frac{3W^{1/3}(U_{0s}+U_{0p})^{1/3}}{2^{1/3}g^{2/3}} \exp\left(\frac{g}{3}z\right). \tag{10}$$

U_{0s} and U_{0p} represent the input energy for s-pulse and p-pulse. W is defined as $W=\beta_{2s}\gamma_s=\beta_{2p}\gamma_p$, which is the width-matching condition to guarantee the self-consistency of the system.P_0 in Eq. (9) is related to the input pulse energies by

$$P_0 = \frac{2^{1/3}g^{2/3}(U_{0s}+U_{0p})^{2/3}}{4W^{1/3}}.$$

. The above theoretical predictions of the incoherent parabolic similariton are confirmed by solving Eqs. (2) and (3), numerically. Two Gaussian pulses with durations of 400 fs for s-pulse and 600 fs for p-pulse are launched into a 20 m long fiber amplifier with the following parameters: $\beta_{2s}=20\ \text{ps}^2\ \text{km}^{-1}$, $\beta_{2p}=24\ \text{ps}^2\ \text{km}^{-1}$, $\gamma_s=0.36\ \text{kW}^{-1}\ \text{m}^{-1}$, $\gamma_p=0.30\ \text{kW}^{-1}\ \text{m}^{-1}$, $g=2\ \text{dBm}^{-1}$, $U_{0s}=0.1$ nJ, and $U_{0p}=0.14$ nJ. The simulation results are summarized in Fig. 1. It can be seen from Figs. 1(c) and 1(d) that, in contrast to a single amplified NLSE, each individual pulse does not evolve into a parabolic pulse. Further simulation beyond 20 m shows that two pulses will propagate self-similarly to maintain the current profiles in Figs. 1(c) and 1(d). It should be noted that the asymptotic shapes of these two similaritons depend on the parameters of the input pulses. We refer to them as component similaritons as distinct from the parabolic similariton obtained from a single amplified NLSE, where the effects of the initial pulse shape and duration gradually wash out along the propagation. The solid curves in Figs. 1(a) and 1(b) denote the corresponding simulated chirp $\omega_j(T)/2\pi$, $(j=s,p)$ compared to the theoretical predictions. Clearly, the good agreement of the simulation results and theory over the central region confirms our conjecture $\omega_j(T)=(g/3\beta_{2j})T$, $(j=s,p)$. T_s and T_p in the figures correspond to the width of the central regions for s-pulse and p-pulse, respectively. The satisfaction of the width-matching condition in the simulation guarantees T_s equal to T_p. The profile of the total power P_{tot}, which is the direct summation of the two curves in Figs. 1(c) and 1(d), is shown in Fig. 1(e). For comparison, the theoretical prediction of Eq. (9) is plotted in the same figure. The excellent agreement verifies the formation of the IPS, which is the asymptotic solution of Eqs. (2) and (3). In contrast to the component similaritons, the IPS retains all the characteristics of the coherent parabolic similariton, such as the independence of the initial pulse shape and duration. Further calculations find that T_s and T_p are equal to the effective width of the parabolic similariton, $T_{\text{pa}}(z)$ defined in Eq. (10).

In order to check the validity of the width matching condition, we performed another numerical experiment changing only three parameters of the above simulation: $\beta_{2p}=20\ \text{ps}^2\ \text{km}^{-1}$, $\gamma_s=0.3\ \text{kW}^{-1}\ \text{m}^{-1}$, and $\gamma_p=0.15\ \text{kW}^{-1}\ \text{m}^{-1}$. In this case, the width-matching condition is violated due to $\beta_{2s}\gamma_s=2\beta_{2p}\gamma_p$. The corresponding chirps shown in

Figs. 2(a) and 2(b) become piecewise linear because of the width mismatch. The effective width of the two pulses T_s and T_p in Figs. 2(c) and 2(d) are also different. Some complicated fine structures appear at the position where the chirps change slope. However, these fine structures will cancel each other when two power profiles in Figs. 2(c) and 2(d) are added together to get the incoherent total power profile, which is shown as the solid curve in Fig. 2(e). It is apparent that this profile can be divided into two regions, each of which is well fit by a parabolic function. Thus, width matching is necessary to form an IPS in that the two pulses stretch at the same speed during the propagation. In contrast, the width mismatch causes the different effective width, which leads to the piecewise linear chirps and piecewise parabolic profiles.

Although only (1+1)-dimensional CNLSEs are discussed in this paper, the concept of the (2+1) or even higher-dimensional IPS can be demonstrated following a similar procedure to that in Ref. [24], where the (2+1)-dimensional coherent parabolic similariton solution was obtained. In addition to the generalization to higher dimensions, the IPS can exist in the more general N-component amplified CNLEs

$$\frac{\partial \varepsilon_j}{\partial \varsigma} + i a_j \frac{\partial^2 \varepsilon_j}{\partial \xi^2} = i \gamma_j \left(\sum_{m=1}^{N} |\varepsilon_m|^2 \right) \varepsilon_j + \frac{g}{2} \varepsilon_j, \tag{11}$$

under the width-matching condition $a_j \gamma_j = c$, $(j=1, \ldots, N)$, where c is a positive constant number. Obviously, Eq. (1) is a special case of Eq. (11). As N goes to infinity, the summation appearing as the coupling is replaced by an integration, which can be used to describe the amplification of a white-light beam [7]. This suggests that it should be possible to generate the white-light IPS.

In conclusion, the incoherent parabolic similariton (IPS) is predicted theoretically in amplified CNLEs under a width-matching condition. In nonlinear pulse propagation, the IPS exists because the nonlinear phase modulation is given by the incoherent sum of the component pulse intensities, so that self-similar propagation occurs with a parabolic intensity profile for the summed pulses, and each component similariton has a linear chirp but generally nonparabolic envelope. In the scenario of spatial beam amplification, a coherent parabolic similariton is guided inside the self-induced waveguide with a parabolic distribution of the refractive index. For an N component incoherent spatial beam, the interplay of the N components through the power coupling forms a multimode waveguide with a parabolic refractive index distribution seen by each individual component beam. Mode beating among them is absent due to the mutual incoherence. Since amplified CNLEs describe many important physical systems, it is anticipated that the IPS is a phenomenon as universal as solitons and will be found in a variety of nonlinear systems incorporating gain.

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